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Equilibrium Shape of Ferrofluid in the Uniform External Field

by Michael Grinfeld and Pavel Grinfeld

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Equilibrium Shape of Ferrofluid Systems in the Uniform External Field

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14. ABSTRACT

It is well known that the morphology of magnetic or electric ferrofluids is extremely sensitive to the applied external electromagnetic fields. Even in the static regimes, they demonstrate a variety of qualitative and quantitative transformations often mentioned as free-surface instabilities. That makes their computational modeling rather challenging. For the sake of validation and verification, there is a need for having exact solutions. Modelers noticed that quite often isolated ferrofluids take on the shape of a spheroid co-axial with the applied quasi-uniform field. There were several claims that a consistent theory allows researchers to prove that an isolated drop of ferrofluid exposed to a uniform field, indeed, has a shape of a spheroid. This report will analyze the computational algorithm that allows us to resolve this and similar problems numerically.

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1. Introduction

Consider a finite chunk of ferrofluid exposed to a uniform magnetic field as shown in Fig. 1.The shape is very close to the shape of a spheroid. Several authors claimed that this fact can be established on the basis of a rigorous theory, like it was done for the equilibrium shape of rotating self-gravitating uniform incompressible liquid (see Chandrasekhar's classics[1] and multiple references therein). To learn more about the magnetic ferrofluids in the uniform magnetic fields, interested readers can find the approaches and further references in the monograph.[2] Other references, relating to ellipsoidal solutions in the problems of electromagnetism, can be found in the works of Stratton,[3] Landau and Lifshitz,[4] and Akhiezer et al.[5]
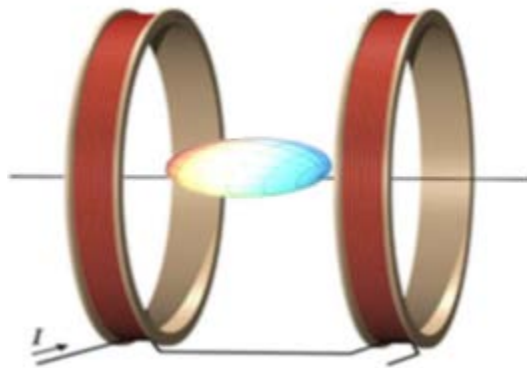
Fig. 1 **The equilibrium shape of a ferrofluid inside the Helmholtz coil takes on a shape close to an elongated spheroid co-axial with the field**

Despite all these remarkable efforts and progress, the problem of equilibrium shape of ferrofluids is far from being exhaustively explored. There are several reasons for further mandatory improvements. For instance, the master systems, describing the dynamics of ferrofluid, face various objections.[6,7] The problem of equations of state—which will be the object of hot debates for a long time—also needs further study. From that standpoint it seems reasonable to limit ourselves to static (thermodynamic) approaches, especially when the problem under study is of a static (equilibrium) nature up front.

There are several fundamental problems relating to the static and thermodynamic aspects of the ponderomotive forces in substances with polar interaction: the authors' vision of those problems has been presented in the works of Michael and Pavel Grinfeld.[8–11] Despite all the existing controversies, the "static" approaches are much older and face less objections than the "dynamics" of electromagnetic media. This report will analyze the problem of equilibrium shape by directly using

the static approach (i.e., not deriving the static system from the more questionable dynamic system of magnetohydrodynamics). When considering the ellipsoidal configurations of ferrofluid we follow the method suggested by Eshelby in the 1950s,[12] instead of the classical approaches of Chandrasekhar,[1] Bashtovoy et al.,[2] Stratton,[3] Landau and Lifshitz,[4] or Akhiezer et al.[5]

2. The Boundary Value Problem to be Solved

We assume that the ferrofluid is maintained at a fixed temperature, and let ψ be the free-energy density at this temperature. Because the fluid is assumed incompressible, the free-energy density is the function of the polarization vector $\vec{P}$, only $\psi = \psi\left(\vec{P}\right)$.

Our suggested approach to this problem is based on the minimization of the total energy of the system W_{total}. The total energy includes 3 ingredients: the total free energy Ψ_{total}, the total magnetic energy E_{mag}, and the total surface energy E_{surf}

$$W_{total} = \Psi_{total} + E_{mag} + E_{surf} , \qquad (1)$$

where the energy ingredients are chosen to be the following:

$$\Psi_{total} = \int\limits_{Body} d\Omega \psi\left(\vec{M}\right), E_{mag} = \int\limits_{Space} d\Omega \left|\vec{H}\right|^2 / 8\pi,$$

$$E_{surf} = \int\limits_{\Xi} d\Omega \sigma\left(\vec{N}\right). \qquad (2)$$

In Eq. 2, $\vec{H}$ is the magnetic field, $\vec{M}$ is the magnetization vector, and $\sigma\left(\vec{N}\right)$ is the dependence of the surface energy density upon the orientation of the unit normal unit.

In the beginning we postulate the classical equations of magnetostatics (when there is no current of free charges, the system of electrostatics and of magnetostatics are the same[4]).

The variational problem under study is the symbiosis of 2 classical problems: the first is the classical problem of magnetostatics and the second is the problem of the equilibrium shape of liquid or crystal. For our situation it is essential that the magnetostatic problem with the given shape of the boundary is essentially linear if the constitutive relationship between the field and inductance is linear. At the same time, the second problem is essentially nonlinear, even when the surface energy σ

is just a constant. The linear electrostatics and magnetostatics can be addressed by various algorithms that have already been developed. Because of its nonlinearity, the problem of the equilibrium shape is treated in the suggested approach as the more difficult. This fact dictates our iterative approach, and we concentrate on this problem in the following paragraphs.

In the first step, we fix the current position of the boundary Ξ and solve the magnetostatic problem for the fixed current domain. This allows us to determine all the current fields $\vec{P}$, $\vec{H}$, and $\vec{B}$. Afterwards, we transition to solving the second problem. Namely, we slightly update the current location of the boundary Ξ in such a way that the total energy W_{total} in the updated domain is smaller than in the current domain. In Eqs. 7–10 we formulate explicitly what should be done to that end. Then, for the updated boundary we return to the first step and repeat the procedure iteratively.

The linear boundary value in the first problem can be solved differently (see the variational approach in Bossavit and Mayergoyz's work[13]). Analytically, it reduces to solving the following system of linear equations and boundary conditions: within the fluid (Eq. 3), within vacuum (Eq.4), at the interface liquid-vacuum (Eq.5) at infinity (Eq. 6), or at a given boundary.

$$\vec{B} = \vec{H} + 4\pi\vec{M}, \ \nabla \cdot \vec{B} = 0, \ \vec{H} = \psi_{\vec{M}}, \tag{3}$$

$$\nabla \cdot \vec{H} = 0, \tag{4}$$

$$\left[\varphi\right]_-^+ = 0, \ \left[\vec{B}\right]_-^+ \cdot \vec{N} = 0, \tag{5}$$

$$\vec{H} \to \vec{H}\infty. \tag{6}$$

In the combined system (Eqs. 3–6), φ is the scalar field potential and $\vec{B}$ is the induction vector, respectively. The symbolic thermodynamic relationship $\vec{H} = \psi_{\vec{M}}$ in Eq. 3 has the following index counterpart: $H_i = \partial\psi\left(M^k\right)/\partial M^i$.

In fact, solving the system (Eqs. 3–6) is equivalent to minimizing the energy W_{total} at fixed boundary Ξ with respect to the distribution of the magnetization vector $\vec{M}$.

The problem of minimizing the total energy W_{total} possess one more essential degree of freedom: it is the position of the boundary Ξ. Theoretical analysis of this

variational problem leads to the boundary equation (presented in the component form)

$$\left[T^{ij}\right]_{-}^{+} N_i N_j + \psi - \frac{\sigma}{R} = \Lambda , \qquad (7)$$

where R is the mean curvature of the boundary, $[a]_{-}^{+}$ symbolizes a jump of the function a across the interface, Λ is an unknown constant (the Lagrange multiplier, which can be determined from the condition of a given total mass of the ferrofluid), and the tensor T^{ij} is defined as follows:

$$T^{ij} = \left(\frac{1}{8\pi} H^k H_k - \frac{1}{4\pi} H^k B_k \right) \delta^{ij} + B^i H^j . \qquad (8)$$

In the empty space, the tensor T^{ij} reduces to the celebrated Maxwell stress tensor.

In the absence of the polarization associated terms, the boundary in Eq. 6 reduces to the classical Laplace condition of a constant mean curvature of the liquid drop under the action of surface tension.

3. A 2-Step Iteration Algorithm

The following statements, established in *Ferroelectrics*,[9] play the key role in the suggested computational algorithm. To formulate them explicitly, consider the current distribution of the magnetization vector $\vec{M}$ satisfying the bulk and boundary equations of magnetostatics (Eqs. 3–6). If we insert this field into the left-hand side of the relationship in Eq. 7 we get a certain function χ, defined on the current surface Ξ. Generally speaking, this function is not constant. If we move each point of the surface in the direction of the local unit normal $\vec{N}$ on the distance Δ such that

$$\Delta = -\tau \left(\chi - \chi_{mean} \right) , \qquad (9)$$

then τ is a sufficiently small positive number and χ_{mean} is the mean value of the of the function χ over the surface Ξ

$$\chi_{mean} = \frac{1}{area\,\Xi} \int_{\Xi} d\Omega . \qquad (10)$$

We call this new surface an upgraded surface Ξ^* and the domain B^* inside it an upgraded domain. Let us solve the magnetostatic problem (Eqs. 3–6) for the

upgraded domain and calculate the upgraded total energy W_{total}^*. The following 2 relationships are valid:

$$W_{total}^* < W_{total} \tag{11}$$

and

$$B^* \cong B , \tag{12}$$

where the symbol $\cong$ means the equality to within the first order of smallness in τ.

4. Toward Verification of the Ellipsoidal Shape

The solution for ellipsoid is of importance for 2 reasons. First, theorists claim[2] that the isolated chunk of ferrofluid in a uniform (at infinity) field takes on the shape of an ellipsoid. Second, this solution being very simple inside the ellipsoid is a convenient tool for quick verification of the computational code. Following Eshelby's research,[12] we will get the presentation of the Cartesian component of the magnetic field outside the ellipsoid as

$$H_i = -A^k \nabla_i \nabla_k \omega + H_i^\infty , \tag{13}$$

where ω is the Newtonian potential of the same ellipsoid having the mass density equal to 1, and A^k has to be determined by the boundary conditions.

The Newtonian potential inside the ellipsoid is very simple:

$$\omega = C_0 - \frac{1}{2} Y_{ij} z^i z^j , \tag{14}$$

where C_0 and Y_{ij} are the constant scalar and symmetric tensor satisfying the conditions

$$Y_{.i}^{i.} = 4\pi . \tag{15}$$

Using Eq. 13, we arrive at the very simple relationship of the magnetostatic field inside the ellipsoid as the following:

$$H_i = A^k Y_{ik} + H_i^\infty . \tag{16}$$

The field outside the ellipsoid is much more complex. Tensor Y_{ij} can be easily expressed in terms of the ellipsoid's semi-axis and orientation and vice versa. When the tensor Y_{ij} is known we can determine the geometry of the ellipsoid.

5. Conclusion

Ferrofluids keep finding more and more applications in engineering practice and academic research. From a scientific point of view, ferrofluids raise many novel and exciting problems in mathematical, theoretical, computational, and applied physics. Many simple fundamental problems of the discipline remain unanswered. Among those is the particularly appealing problem of the equilibrium shape of an isolated chunk of ferrofluid exposed to a static uniform magnetic field. In the suggested approach, this problem is treated as the problem of multidimensional calculus of variation with free (i.e., unknown) boundary. The presence of the unknown boundary makes the problem deeply nonlinear even when the constitutive relationships between the magnetic field and induction are linear.

To solve this and similar problems computationally, we suggested an algorithm similar to the method of steepest descent. The algorithm is based on splitting the solution into 2 steps. The first step consists of solving the linear problem of magnetostatics within the a priori known domain. The second step reduces to updating the current domain in such a way that the total volume of the domain remains unchanged and, at the same time, the total accumulated energy becomes smaller. This step is described by the relationships in Eqs. 7–10. The implementation of this algorithm is currently under development.

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